On Branes and Trace Relations

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2312.00242 JHL
In progress, JHL, D. Stanford

The idea that a large N gauge theory reorganizes into a string theory has a long and fruitful history, dating back from 't Hooft and coming to full-bloom in the AdS/CFT correspondence.

In particular, a state labelled by gauge-invariant words in a large N gauge theory is dual to a state consisting of closed strings propagating in a bulk spacetime:

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A whole lot about the large *N* thermodynamic properties of string theory on AdS backgrounds can be learned from the above statement, especially at weak 't Hooft coupling λ.

[Witten 98, Sundborg 99, Aharony-Marsano-Minwalla-Papadodimas-Van Raamsdonk 03]

This is because, at $\lambda = 0$, the partition function of a $U(N)$ gauge theory counting gauge-invariant states on $S^1_\beta \times S^3$ reduces to a unitary integral over a holonomy matrix U:

$$Z_N(\beta) = \int_{U(N)} dU \, \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(q^n) \, \mathrm{Tr}\, U^n \, \mathrm{Tr}\, U^{-n} \right)$$

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When $f(q)$, the single-letter partition fn, is taken to be that for a superconformal index, these saddles can be interpreted as a sum over BPS families of Euclidean black holes.

Main question

What happens to the statement,

A state labelled by gauge-invariant words in a large N gauge theory is dual to a state consisting of closed strings propagating in a bulk spacetime.

when N takes a finite, integer value?

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In the boundary gauge theory, states formed by exciting the vacuum with single trace operators are orthogonal at large N but become constrained by trace relations when N is finite.

For example, at $N = 2$ a single 2×2 matrix-valued scalar X satisfies the trace identities

$$
\begin{aligned}
&(\mathrm{Tr}\, X)^3 - 3(\mathrm{Tr}\, X)(\mathrm{Tr}\, X^2) + 2(\mathrm{Tr}\, X^3) = 0 \\
&(\mathrm{Tr}\, X)^4 - 6(\mathrm{Tr}\, X)^2(\mathrm{Tr}\, X^2) + 3(\mathrm{Tr}\, X^2)^2 \\
&\qquad\qquad + 8(\mathrm{Tr}\, X)(\mathrm{Tr}\, X^3) - 6(\mathrm{Tr}\, X^4) = 0 \\
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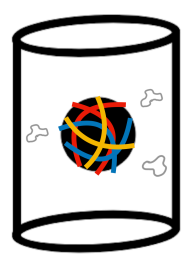
Therefore, the total number of states, at fixed values of the conserved charges and λ, decreases as N is decreased.

On the other hand, in the bulk, non-perturbative states such as black holes and D-branes become lighter at finite g_s.



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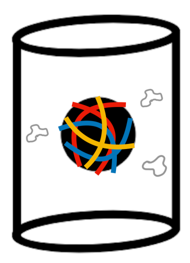

Such objects—which are present in addition to the perturbative string states— would naively result in many more states at finite g_s compared to those at small g_s (at fixed charges and $\lambda = L^4/\alpha'^2$).

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To avoid a paradox, it is typically posited that non-perturbative objects in string theory represent highly redundant descriptions of the same set of quantum states

— so much so that the non-perturbative effects actually need to cut down the total number of states!

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While this principle is thought to be a generic feature of string theory, there is little quantitative understanding of the mechanism underlying the principle.

[Maldacena-Strominger 98, Myers 99, McGreevy-Susskind-Toumbas 00]

Outline

In my talk, I will

(1) describe a non-perturbative effect that implements the stringy exclusion principle in the bulk [JHL-Stanford, in progress]

(2) propose a holographic dual of these non-perturbative contributions in the boundary gauge theory [JHL 23]

Giant graviton expansion

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$Z_\infty(q)$: index of S^5 Kaluza-Klein modes

$q^{kN} \hat{Z}_k(q)$: index of k giants and their quantum fluctuations

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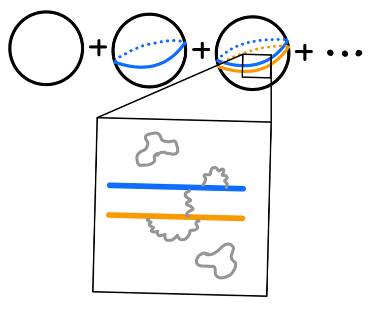

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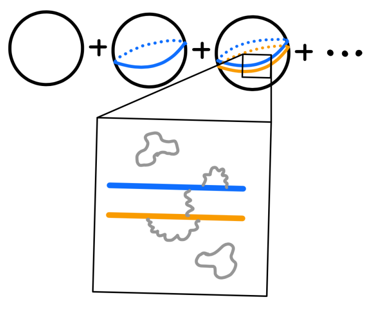
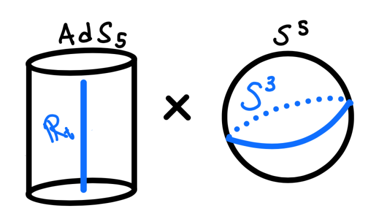


[R Arai, M Beccaria, A Cabo-Bizet, S Choi, D S Eniceicu, S Fujiwara, D Gaiotto, F F Gautason, H Hayashi, Y Imamura, S Kim, E Lee, JHL, J Lee, J Liu, T Mori, S Murthy, T Nosaka, T Okazaki L Pando Zayas, A Tseytlin, J van Muiden, ...]

Giant gravitons are BPS branes on $\mathbb{R} \times S^3 \subset \mathrm{AdS}_5 \times S^5$ that are supported via a large angular momentum induced from the background Ramond-Ramond potential C_4.

[McGreevy-Susskind-Toumbas 00]



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But the giant graviton expansion adds a new twist to this story.

Half-BPS example

To understand properties of the expansion, let's examine the formula in the simplest, half-BPS example:

$$\frac{1}{\prod_{n=1}^{N}(1-q^n)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=0}^{\infty} (-1)^k q^{kN} \frac{q^{k(k+1)/2}}{\prod_{m=1}^{k}(1-q^m)}.$$

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The half-BPS sector of $U(N)$ $\mathcal{N}=4$ SYM is a purely bosonic sector consisting of the states

$$\prod_i \mathrm{Tr}\, X^{n_i} |0\rangle,$$

modulo trace relations at finite N. LHS is the spectrum of these gauge invariants.

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On the other hand, the bulk spectrum seems to overcount the half-BPS spectrum of $U(N)$ $\mathcal{N}=4$ SYM:

$$\begin{aligned}\mathrm{RHS} = 1 &+ q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots \\ &+ q^N \left(-q - 2q^2 - 4q^3 - 7q^4 - 12q^5 - 19q^6 - \cdots\right) \\ &+ q^{2N} \left(q^3 + 2q^4 + 5q^5 + 9q^6 + 17q^7 + 28q^8 + \cdots\right) \\ &+ q^{3N} \left(-q^6 - 2q^7 - 5q^8 - 10q^9 - 19q^{10} - 33q^{11} - \cdots\right)\end{aligned}$$

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Here, the stringy exclusion principle at finite N is a consequence of the fact that there are delicate cancellations in the bulk spectrum, as opposed to a hard cutoff.

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Recall that the half-BPS sector is a purely bosonic sector, so $(-1)^k$ could not have been due to the fermion number operator $(-1)^F$ counting *physical* fermions mod $\mathbb{Z}_2$.

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Surprisingly, we will find that $(-1)^k$ has a purely bosonic origin.

Brane path integral

Let us compute the contribution of a wrapped D3-brane to the half-BPS partition function

$$Z_{\frac{1}{2}\text{-BPS}}(q) = \lim_{\beta\to\infty} \mathrm{Tr}_{\mathcal{H}_{S^3}} \left(e^{-\beta(H-R)} q^R \right).$$

$Z_{\frac{1}{2}\text{-BPS}}$ must be defined in terms of a projection as there is no trace over $\mathcal{H}_{S^3}$ of $U(N)$ $\mathcal{N}=4$ SYM that gives $Z_{\frac{1}{2}\text{-BPS}}$.

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In the half-BPS sector, the index and partition functions are equivalent up to an overall Casimir energy.

To this end, we consider a thermal $AdS_5 \times S^5$ twisted by a background chemical potential μ for the relevant R-charge R:

$$\begin{aligned} ds^2 = & \overbrace{\left(1+\rho^2\right) d\tau^2 + \frac{1}{1+\rho^2} d\rho^2 + \rho^2 d\hat{\Omega}^2_{S^3}}^{EAdS_5} \\ & \underbrace{+\frac{1}{1-r^2} dr^2 + r^2 \left(d\theta + i(\mu-1)d\tau\right)^2 + (1-r^2) d\Omega^2_{S^3}}_{S^5} \\ C_4 = & -i\rho^4 d\tau \wedge d\hat{\Omega}_{S^3} + (1-r^2)^2 \left(d\theta + i(\mu-1)d\tau\right) \wedge d\Omega_{S^3}, \end{aligned}$$

where $\tau \sim \tau + \beta$ and $q = e^{-\beta\mu}$.

[Beccaria-Cabo-Bizet 24] [JHL-Stanford, In progress]

The euclidean continuation of the D3 giant graviton is wrapped on the $S^1_\beta \times S^3$ of the twisted $AdS_5 \times S^5$. The brane solution sits at the center of AdS_5 and wraps a maximal $S^3 \subset S^5$.

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We will evaluate the partition function of this brane

$$\widehat{Z}_1 = \int D\Phi \, e^{-S_{D3}[\Phi]},$$

with bosonic action

$$S_{D3} = \frac{N}{2\pi^2} \int \left(d\tau d\Omega \sqrt{\det g_{D3}} - iP[C_4] \right)$$

to 1-loop order.

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To 1-loop, the half-BPS sector receives non-trivial contributions only from two scalars $\phi, \bar{\phi}$ that parametrize the transverse fluctuations of the brane in $S^3 \subset S^5$:

$$S_{\mathrm{quad}} = \frac{1}{2\pi^2} \int d\tau d\Omega_{S^3} \left[\dot{\bar{\phi}}\dot{\phi} + \partial_i \bar{\phi} \partial^i \phi + (1+\mu) \left(\bar{\phi}\dot{\phi} - \dot{\bar{\phi}}\phi \right) + \left(1 - (1+\mu)^2 \right) \bar{\phi}\phi \right]$$

$$= \sum_{n\in\mathbb{Z}} \sum_{k=0}^{\infty} \sum_{r_1, r_2 = -\frac{k}{2}}^{\frac{k}{2}} \bar{\phi}_n^{(k,r_1,r_2)} \left[\left(\frac{2\pi n}{\beta} \right)^2 + (1+\mu) \left(\frac{4\pi i n}{\beta} \right) - (1+\mu)^2 + (k+1)^2 \right] \phi_n^{(k,r_1,r_2)}$$

Until this point, the analysis is entirely standard.

But let's take a closer look at the lowest harmonics on $S^1_\beta \times S^3$:

$$S_{quad} = -\mu(\mu+2)\,\bar{\phi}_0^{(0,0,0)}\phi_0^{(0,0,0)} + (higher\, modes)$$

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In the 1-loop path integral, the contour rotation introduces a factor of i for each unstable mode. Since there are two such modes, $\widehat{Z}_{1-loop}$ for a single brane has an overall minus sign.

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Why should the rotated contour give the correct contribution to the defining contour of the problem? i.e. in what situations does a bosonic saddle contribute with a minus sign?

To understand when this happens, consider a toy integral

$$I = \frac{1}{2\pi} \int d\theta d\phi \, \sin\theta \, e^{-\beta(1-\cos\theta)} \; = \; \frac{1}{\beta}\left(1 - e^{-2\beta}\right)$$
$$= \frac{1}{2\pi} \int d\theta d\phi dc_\phi dc_\theta \, e^{-\beta(1-\cos\theta)+\sin\theta c_\theta c_\phi}$$

which appears as a prototypical example for the Duistermaat-Heckman localization formula.

The integral is invariant under the hidden supersymmetry

$$Q\theta = c_\theta, \qquad Q\phi = c_\phi, \qquad Qc_\theta = 0, \qquad Qc_\phi = -\beta.$$

Therefore, we can represent Q as

$$Q = d + \iota_V$$

where $V = -\beta \frac{\partial}{\partial \phi}$ and it holds that $Q^2 = \mathcal{L}_V$. The action is Q-exact and the integral localizes to fluctuations around fixed points $\theta = 0, \pi$ of the vector field V.

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The integral I can be exactly evaluated both directly and in the saddle approximation around the north and south poles.

North pole: (dominant saddle) Take $\theta = \widehat{\theta}$ with small $\widehat{\theta}$. Then

$$Z^{(N)} = \int d\widehat{\theta}\, dc_\phi\, dc_{\widehat{\theta}}\; e^{-\frac{1}{2}\beta\widehat{\theta}^2 + \widehat{\theta}\, c_{\widehat{\theta}}\, c_\phi}$$

$$= \int_0^\infty d\widehat{\theta}\; \widehat{\theta}\;\; e^{-\frac{1}{2}\beta\widehat{\theta}^2} \; = \; \frac{1}{\beta}.$$

The saddle at the north pole is the dominant saddle, but it overestimates the answer.

South pole (subleading saddle): . Take $\theta = \pi - \widehat{\theta}$ with small $\widehat{\theta}$. Then

$$Z^{(S)} = e^{-2\beta} \int d\widehat{\theta}\, dc_\phi\, dc_{\widehat{\theta}}\; e^{\frac{1}{2}\beta\widehat{\theta}^2 - \widehat{\theta}\, c_{\widehat{\theta}}\, c_\phi}$$

$$= e^{-2\beta} \int_{\pm i\infty}^{0} d\widehat{\theta}\; (-\widehat{\theta})\; e^{\frac{1}{2}\beta\widehat{\theta}^2} \; = \; -\frac{e^{-2\beta}}{\beta}.$$

There is an overall minus sign from the contour rotation.

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Let us decompose the defining contour for θ into Lefschetz thimbles associated with the critical points at the north and south poles. We can wiggle the steepest descent contour emanating from $\theta = 0$ by introducing a small positive imaginary part to β.

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This explains the range of integration for $\widehat{\theta}$ in $Z^{(S)}$.

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So we can interpret minus sign in the brane path integral as that compensating for the overcounting of the naive half-BPS Kaluza-Klein spectrum: $\frac{1}{\prod_{n=1}^{\infty}(1-q^n)}$

With this lesson in mind, let us return to the computation of the 1-loop brane partition function.

The 1-loop brane partition function with all fluctuations is

$$Z_{1-\text{loop}}\left[p=e^{-\beta}\right]=$$

$$\prod_{n=0}^{\infty}\overbrace{\left[\frac{1}{(1-qp^{n+2})(1-q^{-1}p^{n})}\right]^{(n+1)^2}}^{S^5\text{ Scalars}}\overbrace{\left[\frac{1}{1-p^{n+1}}\right]^{4(n+1)^2}}^{\text{AdS}_5\text{ Scalars}}\overbrace{\left[\frac{1}{1-p^{n+2}}\right]^{2(n+1)(n+3)}}^{\text{Vector}}$$

$$\times\underbrace{\left[\left(1+\sqrt{q}p^{n+2}\right)\left(1+\frac{1}{\sqrt{q}}p^{n+2}\right)\left(1+\sqrt{q}p^{n+1}\right)\left(1+\frac{1}{\sqrt{q}}p^{n+1}\right)\right]^{2(n+1)(n+2)}}_{\text{Fermions}}$$

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The 1-loop analysis may be generalized to k-coincident wrapped branes, and their BPS spectrum is

$$(-1)^k q^{kN} \frac{q^{k(k+1)/2}}{\prod_{m=1}^{k}(1-q^m)}.$$

There are $2k$ gauge-invariant, unstable bosonic modes on their worldvolumes.

So far, we've seen that BPS partition functions $\widehat{Z}_k(q)$ of k giant graviton branes possess overall signs $(-1)^k$, which resulted from the fact that there were $2k$ unstable bosonic scalar modes on the worldvolume.

Now suppose we wish to associate a state-counting interpretation

$$\mathrm{Tr}_{\mathcal{H}_N} e^{-\mu R}$$

for the sum

$$\frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=0}^{\infty} (-1)^k q^{kN} \frac{q^{k(k+1)/2}}{\prod_{m=1}^{k}(1-q^m)}$$

over $\widehat{Z}_k(q)$. Then $(-1)^k$ would need to come from a grading in the associated Hilbert space $\mathcal{H}_N$.

However, we know that the half-BPS Hilbert space $\mathcal{H}_N$ in $\mathcal{N} = 4$ SYM is purely bosonic and is graded only with respect to the energy H and the R-charge R.

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In other words, there seems to be no extra grading required to provide the $(-1)^k$.

Obvious question: How is the BPS spectrum $\widehat{Z}_k$ of D3 giants in $\mathrm{AdS}_5 \times S^5$ encoded in the boundary $U(N)$ $\mathcal{N} = 4$ SYM?

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$$q^{kN} \frac{q^{k(k+1)/2}}{\prod_{n=1}^{k}(1-q^n)}.$$

Identifying such states is made subtle because, as argued above, these states cannot belong in the physical half-BPS Hilbert space $\mathcal{H}_N$ of the gauge theory.

We obtain clues regarding the nature of these “states” by examining the “mode” expansion in z of the determinant operator $\det(z - X)$ inserted at a point in $U(N)$ $\mathcal{N} = 4$ SYM.

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This operator is dual to a family of D3 giant graviton solutions wrapped on $S^3 \subset S^5$ with order N units of an R-charge.

The auxiliary parameter z has an interpretation as the bulk insertion point of the brane on the plane transverse to S^3 in S^5.

The fluctuation “modes” in z of the determinant operator are

$$\det(z - X) = z^N e^{-\sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \operatorname{Tr} X^n} = z^N \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} P_n(\operatorname{Tr} X^\bullet).$$

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The first few coefficients are

$$\begin{aligned}
P_1 &= \operatorname{Tr} X \\
P_2 &= \frac{1}{2}\Big((\operatorname{Tr} X)^2 - (\operatorname{Tr} X^2)\Big) \\
P_3 &= \frac{1}{6}\Big((\operatorname{Tr} X)^3 - 3(\operatorname{Tr} X)(\operatorname{Tr} X^2) + 2(\operatorname{Tr} X^3)\Big) \\
P_4 &= \frac{1}{24}\Big((\operatorname{Tr} X)^4 - 6(\operatorname{Tr} X)^2(\operatorname{Tr} X^2) + 3(\operatorname{Tr} X^2)^2 \\
&\qquad + 8(\operatorname{Tr} X)(\operatorname{Tr} X^3) - 6(\operatorname{Tr} X^4)\Big)
\end{aligned}$$

and so on.

At any positive integer value N, the tower of "modes"

$$\det(z - X) = (-1)^{N+1}\left[\cdots + \frac{P_{N+1}}{z} - \frac{P_{N+2}}{z^2} + \frac{P_{N+3}}{z^3} + \cdots\right]$$

starting at P_{N+1} are but the trace relations between gauge-invariant operators that are present in any $U(N)$ gauge theory with the complex scalar X.

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In other words, states constructed by acting with modes $P_{N+1}, P_{N+2}, \cdots$ on the vacuum $|0\rangle$ are the finite N null states.

These are null states that need to be separately accounted for only if we decide to work with the gauge-invariant variables

$$\mathrm{Tr}\, X, \mathrm{Tr}\, X^2, \mathrm{Tr}\, X^3, \cdots$$

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rather than with the matrix components.

We observe that a single tower of these null states have the correct BPS spectrum

$$q^N \frac{q}{1-q}$$

of a single D3 giant.

Let us account for the null state constraints $P_{N+1}, P_{N+2}, \cdots$ using the BRST formalism.

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To impose constraints in the BRST formalism, we introduce anti-commuting ghosts χ along with a homological charge $\widehat{Q}$ acting as

$$[\chi_{-N-a}, \widehat{Q}]_{\pm} = P_{N+a}$$
$$[P_a, \widehat{Q}]_{\pm} = 0,$$

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where $a = 1, 2, \cdots$. The (co)homology of $\widehat{Q}$ is the half-BPS state space of $U(N)$ $\mathcal{N} = 4$ SYM, which is the quotient of the half-BPS state space of $U(\infty)$ SYM by the trace relations.

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Assign ghost number 1 to χ_a and -1 to the differential $\widehat{\mathcal{Q}} = [\,\cdot\,, \widehat{Q}]_{\pm}$.

These auxiliary ghosts should be distinguished from the usual BRST (anti-)ghosts that one introduces for gauge-fixing in a $U(N)$ gauge theory.

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Rather, the role of the auxiliary ghost χ_a and the differential $\widehat{Q}$ is to supplement the $U(\infty)$ theory with directions for systematically removing all but the $U(N)$ degrees of freedom.

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The states with ghost number k are multi-ghost states built out of k χ's. We can write a basis of k multi-ghost states as

$$\chi_{-N-a_1}\chi_{-N-a_2}\cdots\chi_{-N-a_k}|0\rangle$$

where $1 \leq a_1 < a_2 < \cdots < a_k < \infty$, which yields the charge spectrum

$$(-1)^k q^{kN}\frac{q^{k(k+1)/2}}{\prod_{n=1}^{k}(1-q^n)}.$$

This agrees with the bulk computation of the BPS spectrum of k D3 giants. Here, the signs $(-1)^k$ *must* be included because the ghosts χ are implementing null relations.

What does the identification between the half-BPS states of bulk D-branes and auxiliary ghosts in a (suitably-rewritten) boundary $\mathcal{N} = 4$ SYM suggest about the bulk half-BPS space of states?

We can organize the space of multi-ghost states, with coefficients in the ring of polynomials of traces

$$R = \mathbb{C}[\mathrm{Tr}\, X, \mathrm{Tr}\, X^2, \mathrm{Tr}\, X^3, \cdots],$$

into a complex

$$\cdots \to \mathcal{V}_3 \overset{\widehat{Q}}{\to} \mathcal{V}_2 \overset{\widehat{Q}}{\to} \mathcal{V}_1 \overset{\widehat{Q}}{\to} \mathcal{H}_\infty \to 0,$$

where $\mathcal{V}_k$ is the space of multi-ghost states of ghost number k, with coefficients in the ring R.

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$\mathcal{H}_\infty = R|0\rangle$: half-BPS space of states of $\mathcal{N} = 4$ SYM at infinite N.

$\widehat{\mathcal{Q}} = [\,\cdot\,, \widehat{Q}]_\pm$ acts on operators via the graded Leibniz rule.

$$\cdots \to \mathcal{V}_3 \xrightarrow{\widehat{Q}} \mathcal{V}_2 \xrightarrow{\widehat{Q}} \mathcal{V}_1 \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \to 0,$$

The comparison of the spectrum suggests that $\mathcal{V}_k$ should be identified with the perturbative space of states built on $\mathrm{AdS}_5 \times S^5$ in the presence of k-coincident half-BPS giants.

$$\cdots \to \mathcal{V}_3 \xrightarrow{\widehat{Q}} \mathcal{V}_2 \xrightarrow{\widehat{Q}} \mathcal{V}_1 \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \to 0,$$

This complex has non-vanishing homology only at zero ghost number. Since the image $\widehat{Q}\mathcal{V}_1 \subset \mathcal{H}_\infty$ is the space of trace relations with coefficients in R, the homology at ghost number $k = 0$ is the Hilbert space

$$\mathcal{H}_N = \mathcal{H}_\infty / \widehat{Q}\mathcal{V}_1.$$

in the half-BPS sector of $U(N)$ $\mathcal{N} = 4$ SYM at finite N.

Let us observe that each $\mathcal{V}_k$, $k = 0, 1, 2, \cdots$ in the above complex is a Fock space built from the multi-ghost generators

$$\chi_{-N-a_1}\chi_{-N-a_2}\cdots\chi_{-N-a_k}|0\rangle$$

with coefficients in $R = \mathbb{C}[\mathrm{Tr}\,X, \mathrm{Tr}\,X^2, \mathrm{Tr}\,X^3, \cdots]$.

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with coefficients in $R = \mathbb{C}[\mathrm{Tr}\,X, \mathrm{Tr}\,X^2, \mathrm{Tr}\,X^3, \cdots]$.

(i.e. $\mathcal{V}_k$ are free R-modules)

This means that, if we augment the complex with $\mathcal{H}_N$ at the final step, the resulting exact sequence

$$\cdots \to \mathcal{V}_3 \xrightarrow{\widehat{Q}} \mathcal{V}_2 \xrightarrow{\widehat{Q}} \mathcal{V}_1 \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \to \mathcal{H}_N \to 0$$

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is a “free resolution” of the half-BPS space of states $\mathcal{H}_N$ at finite N.

Roughly speaking, a free resolution is the procedure of replacing a highly-constrained space $\mathcal{H}_N$ with a sequence of unconstrained Fock spaces $\mathcal{V}_k$, whose generators represent the constraints on $\mathcal{H}_N$.

The resolution gives a precise relationship between

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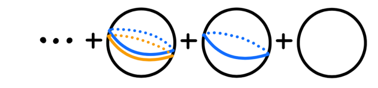
identified as the *bulk* half-BPS space of states in $\mathrm{AdS}_5 \times S^5$.



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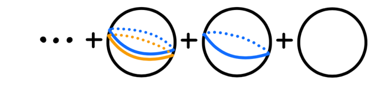

This is an extended Hilbert space whose (co)homology yields the physical Hilbert space $\mathcal{H}_N$.

The resolution says that quantum flucations of certain bulk D-branes are encoded in the *relations* between the single-trace generators that arise in the boundary gauge theory at finite N.

It is simple to use this observation to recover the giant graviton expansion

$$\frac{1}{\prod_{n=1}^{N}(1-q^n)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=0}^{\infty}(-1)^k q^{kN} \frac{q^{k(k+1)/2}}{\prod_{m=1}^{k}(1-q^m)}.$$

in the half-BPS sector.

It is a property of exact sequences, e.g.

$$\cdots \to \mathcal{V}_3 \xrightarrow{\widehat{Q}} \mathcal{V}_2 \xrightarrow{\widehat{Q}} \mathcal{V}_1 \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \to \mathcal{H}_N \to 0$$

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the alternating sum of the dimensions of spaces in the complex vanishes.

Therefore, we can express the charge spectrum of $\mathcal{H}_N$ in terms of an alternating sum of the charge spectra of $\mathcal{V}_k$ over $k = 0, 1, 2, \cdots$:

$$\mathrm{Tr}_{\mathcal{H}_N} q^R = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k} q^R.$$

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This alternating sum formula shares its origin with that appearing in the BRST approach to minimal models on a torus. The space $\mathcal{V}_k$ of k-th order relations in our context plays the role of the space of Virasoro descendants of singular vectors in 2d CFTs.

The alternating sum formula yields the half-BPS giant graviton expansion:

$$\frac{1}{\prod_{n=1}^{N}(1-q^n)} = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=0}^{\infty}(-1)^k q^{kN} \frac{q^{k(k+1)/2}}{\prod_{m=1}^{k}(1-q^m)}.$$

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The closed string spectrum $Z_\infty = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$ factored out on the RHS, because $\mathcal{V}_k$ are free R-modules and Z_∞ is the half-BPS partition function of $\mathcal{H}_\infty = R|0\rangle$.

What is the bulk interpretation of the differential $\widehat{Q}$?

What is the bulk interpretation of the differential $\widehat{Q}$?

It maps a half-BPS state of k coincident D3 giants to linear combinations of states of $k-1$ coincident D3 giants with coefficients in the traces (i.e. closed strings), e.g.

$$\widehat{\mathcal{Q}}\chi_{-N-a} = P_{N+a}$$
$$\widehat{\mathcal{Q}}(\chi_{-N-a}\chi_{-N-b}) = P_{N+a}\,\chi_{-N-b} - P_{N+b}\,\chi_{-N-a}$$

where $P_{N+a} = P_{N+a}(\mathrm{Tr}\,X^{\bullet})$ is a trace relation of charge $N+a$.

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where $P_{N+a} = P_{N+a}(\mathrm{Tr}\,X^{\bullet})$ is a trace relation of charge $N+a$.

Thus a natural interpretation for $\widehat{Q}$ is that it is an instanton that interpolates between a pair of vacua labelled by k and $k-1$ wrapped D-branes in $\mathrm{AdS}_5 \times S^5$.

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2. This complex furnishes a BRST-like resolution of the finite N space of states $\mathcal{H}_N$ of the boundary gauge theory

While the bulk computations in the half-BPS sector relied on supersymmetry and the presence of a weakly-curved gravity limit, let me argue that the lessons above do not rely crucially on those properties.

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Trace relations between gauge-invariant operators are present in any finite N gauge theory with adjoint fields.

While the bulk computations in the half-BPS sector relied on supersymmetry and the presence of a weakly-curved gravity limit, let me argue that the lessons above do not rely crucially on those properties.

Trace relations between gauge-invariant operators are present in any finite N gauge theory with adjoint fields.

At least when $\lambda = 0$, it is always possible to write the space of states of this theory as that of a $U(\infty)$ gauge theory supplemented with auxiliary ghosts for trace relations that would have been present at a value of N.

In more general situations, e.g. when $\mathcal{H}_\infty = R|0\rangle$ is given by

$$R = \mathbb{C}[\mathrm{Tr}\, X, \mathrm{Tr}\, XY, \mathrm{Tr}\, \psi F..., \mathrm{Tr}\, \partial...\partial...X, \cdots].$$

acting on the vacuum $|0\rangle$, the trace relations may satisfy non-trivial relations among themselves, e.g.

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Then computing a free resolution requires introducing ghosts-for-ghosts, etc. as in the BRST formalism. The algorithm for computing such a resolution is called the *Koszul-Tate resolution.* [Tate 57, Henneaux-Teitelboim 92]

Lefschetz trace formula

Assuming that one has computed the Koszul-Tate resolution (up to an energy cutoff),

$$\cdots \to \mathcal{V}_3 \xrightarrow{\widehat{Q}} \mathcal{V}_2 \xrightarrow{\widehat{Q}} \mathcal{V}_1 \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \to \mathcal{H}_N \to 0$$

the Lefschetz trace formula relates the trace taken over $\mathcal{H}_N$ with an alternating sum of the traces taken over $\mathcal{V}_k$.

For instance, the free thermal partition function of a $U(N)$ gauge theory on $S^1 \times S^3$ is equal to the alternating sum of the free thermal partition functions over $\mathcal{V}_k$ of the Koszul-Tate resolution:

$$\mathrm{Tr}_{\mathcal{H}_N} e^{-\beta H} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k} e^{-\beta H}.$$

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$$\mathrm{Tr}_{\mathcal{H}_N} e^{-\beta H} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k} e^{-\beta H}.$$

The thermal correlation functions are related as

$$\begin{aligned} &\mathrm{Tr}_{\mathcal{H}_N} \left[e^{-\beta H}\, \mathcal{O}_1(\tau_1, x_1)\mathcal{O}_2(\tau_2, x_2) \cdots \mathcal{O}_n(\tau_n, x_n) \right] \\ &\qquad = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k} \left[e^{-\beta H}\, \mathcal{O}_1(\tau_1, x_1)\mathcal{O}_2(\tau_2, x_2) \cdots \mathcal{O}_n(\tau_n, x_n) \right] \end{aligned}$$

where $0 \leq \tau_n < \cdots < \tau_2 < \tau_1 < \beta$.

For superconformal gauge theories with non-anomalous R-symmetry, we can write in addition the relation between the superconformal indices

$$\mathrm{Tr}_{\mathcal{H}_N}\left[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q_1^{\mathcal{C}_1} q_2^{\mathcal{C}_2}\cdots\right] = \sum_{k=0}^{\infty}(-1)^k \mathrm{Tr}_{\mathcal{V}_k^N}\left[(-1)^F e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}} q_1^{\mathcal{C}_1} q_2^{\mathcal{C}_2}\cdots\right]$$

where $\mathcal{C}_i$ are conserved charges that commute with the supercharge $\mathcal{Q}$.

Therefore, the Lefschetz trace formula appears to provide a gauge-string map at $\lambda = 0$ and finite N.

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The bulk observables are computed by taking the alternating sum of the expectation values in an ensemble of states built on each open string vacuum.

Currently, the tools available in the bulk only allow us to compute BPS quantities in the large λ regime and then to extrapolate those quantities to $\lambda \to 0$ for comparison to gauge theory computations.

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It would be nice to compare the gauge and string theory results directly at $\lambda = 0$ in the future, e.g. conformal boundary conditions of tensionless strings in $\mathrm{AdS}_5 \times S^5$. [Gaberdiel-Gopakumar 21]

Thank you